Eur. Phys. J. B **50**, 215–220 (2006)
DOI: 10.1140/epjb/e2006-00145-6

THE EUROPEAN PHYSICAL JOURNAL B

A selfsimilar behavior of the urban structure in the spatially inhomogeneous model

E.Y. Echkina, O.I. Inovenkov[a], and D.P. Kostomarov

Moscow State University, Moscow, Russia

Received 30 September 2005 / Received in final form 3 December 2005
Published online 12 April 2006 –

Abstract. At present there is a strong tendency to use new methods for the description of the regional and spatial economy. In increasing frequency we consider that any economic activity is spatially dependent. The problem of the evolution of internal urban formation can be described with the exact supposition. So that is why we use partial derivative equations set with the appropriate boundary and initial conditions for the solving the problem of the urban evolution. Here we describe the model of urban population's density modification taking into account a modification of the housing quality. A program has been created which realizes difference method of mixed problem solution for population's density. For the wide class of coefficients it has been shown that the problem's solution "quickly forgets" the parts of the initial conditions and comes out to the intermediate asymptotic form, which nature depends only on the problem's operator. Actually it means that the urban structure does not depend on external circumstances and is formed by the internal structure of the model.

PACS. 89.65.Gh Economics; econophysics, financial markets, business and management – 89.65.Lm Urban planning and construction

1 Introduction

All sorts of economical activity is brought into correlation with the certain place and time, thus while considering evolutional systems we must take into account the spatial dependence. Due to the progress in transportation and communication facilities the interaction between different economical agents is getting more and more dependent on their location. It is very important to realize and to describe characteristics of these spatial interactions. As a consequence of technological progress and changes in people's behavior urban problems have become more complicated. Nowadays, the increasing spatial and time variety of the passing urban processes characterize urban systems. Such metropolises like New York, Stockholm, Paris and Tokyo have the most complicated urban structure. Urban centralization can be found either in highly developed or in less developed countries, but in highly developed countries decentralization processes have become apparent. In particular European Union has a great influence upon these processes. There are a lot of geographical and urban models, which describe current and future urban processes. Nowadays there are three basic approaches. The first one is called neoclassical urban economy. Urban economists developed it in the end of 19th century. Since 1860-s a lot of models have appeared. But this approach is seriously limited by the analysis of the equilibrium positions. In the second approach (e.g. Wilson) time and place are very important. However as the area is divided into discrete zones, there is no chance to explain inner structure of urban areas. The third approach is called "spatial dynamic approximation". It is using continuous area for the solving dynamic problems. Beckman and Puu based upon von Tuhnen's method. They supposed that economical activity is spatially dependent by itself [1–3]. The spatial density describes this activity. So the problem of urban evolution is described by the partial derivative equations with the corresponding initial and boundary conditions.

2 The model of urban system considering the spatial diffusion

Let's describe a modification model [4] of the frequency distribution of the urban population in the urban area. We assume a modification of the housing quality. Let us consider heterogeneous urban system, which is characterized by the functions below:

$u(x,y,t)$ — population density in the point $M(x,y)$ at time t,
$q(x,y,t)$ — housing quality (cost of realty) in the point,
$M(x,y)$ at time t,

where $r=\sqrt{x^2+y^2}$ is a destination between the "downtown" and the place of residence.

We do not take into account the demographical and migration processes between the city and neighbor regions. Considering urban population's diffusion we can

[a] e-mail: inovenkov_oleg@yahoo.com

describe urban system as follows:

$$\begin{cases} u_t = \alpha(f(q) - u) + \mathrm{div}(\theta \mathrm{grad}\, u), \\ q_t = -\delta q + H(I(i, q)), \\ M(x, y) \in G. \end{cases} \quad (1)$$

where G is a concerned region of urban space, α — adaptation parameter, θ — population's diffusion coefficient, δ — housing fracture velocity. Note that coefficient θ can depend on u and q, and also on x, y and t. We do not consider a diffusion modification of the housing quality, so the equation for q is an ordinary differential one. Note that one-dimensional variant of this model was suggested by Zhang.

For the u function we have a boundary condition of a third kind at the boundary of G area:

$$\theta \frac{\partial u}{\partial n} + \beta u|_\partial G = 0, \quad (2)$$

if $\beta = 0$, then boundary condition

$$\frac{\partial u}{\partial n}|_{\partial G} = 0 \quad (3)$$

means that during concerned period there is no changes in population. If $t = 0$ we have additional initial conditions for u and q:

$$\begin{cases} u|_{t=0} = u_0(x, y), \\ q|_{t=0} = q_0(x, y). \end{cases} \quad (4)$$

Eventually the second equation in system (1) describes modification of the housing quality. Term δq with minus, which is a part of this equation, describes housing destruction effects. We consider that the owners who define the quantity of expenses maintain housing conditions. So the housing price depends on the rent revenue of the owner. Let the total income be defined as I. The income in the fixed point depends on the density of population and housing quality, that is $I = I(u, q)$. The derivative $\partial I/\partial u$ of this functional has no definite sign and the derivative $\partial I/\partial q$ is above zero. Under the constant level of q the sign of $\partial I/\partial u$ generally is not definite because increase or decrease of the income depends on the current situation. Derivative $\partial I/\partial q$ is above zero because housing quality improvement under fixed level of population density has to lead to the increasing of owner's income. We suppose that housing maintenance expenses are positively connected with the income, that is $dH/dI > 0$. Simply we define expenditure function for the maintenance expenses as follows:

$$H(I) = \frac{\mu u q^2}{1 + \sigma u} \quad (5)$$

where μ and σ are positive coefficients. If we interpret

$$\frac{q^2}{1 + \sigma u} \quad (6)$$

as a housing rent, then

$$\frac{u q^2}{1 + \sigma u} \quad (7)$$

will be total income of the housing owner into the current point. Parameter μ can be considered as a ratio of maintenance expenses and total income.

As a first step of this model research we limit ourselves with the one-dimensional analysis when functions u and q depends on r and t, where $r = \sqrt{x^2 + y^2}$ is a destination between the "down-town" and the place of residence.

Let us make all the variables dimensionless corresponding the formulas below:

$$\alpha t \to t, q = \frac{\mu Q}{\alpha \sigma}, u = \frac{U}{\sigma},, k = \frac{\theta}{\alpha}, \nu = \frac{\delta}{\alpha},$$
$$g(Q) = \sigma f\left(\frac{\mu Q}{\alpha \sigma}\right). \quad (8)$$

Then the mathematical problem of determination U and Q functions will be:

$$\begin{cases} \frac{\partial U}{\partial t} = g(Q) - U + \frac{1}{r}\frac{\partial}{\partial r} k(U, Q, r, t) r \frac{\partial U}{\partial r}, \\ \frac{\partial Q}{\partial t} = -\nu Q + \frac{U Q^2}{1+U}, \\ k(U, Q, r, t) r \frac{\partial U}{\partial t} + \beta U|_{r=1} = 0, \\ U|_{t=0} = U_0(r), \\ Q|_{t=0} = Q_0(r). \end{cases} \quad (9)$$

The question concerning unique existence of problem's solution is not very simple but for the wide class of nonlinear coefficients unique existence theorems of the classical problem's solution are still valid.

It seems very interesting to study a solution's behavior of the system (9) against initial conditions, especially the existence of the intermediate asymptotic forms. They exist when the solution "quickly forgets" the parts of the initial conditions and develops according to the internal structure of the model.

For the computational solution of the system (9) we take difference scheme with the second level of the approximation upon the radius and with the first level upon the time.

2.1 Diagnostics of the computational experiment

For the software implementation of the problem solution we choose C++Builder. The special dialogue interface has been created with the dialogue boxes which let input initial conditions: initial and boundary conditions,number of points N, in which we divide segment $[0, 1]$, time step τ, bifurcation parameter ν, coefficient β in the boundary condition of the third level. Diagrams show dependence of the final solution during the last time layer on the destination between the "down-town" and the place of residence. The first line is demonstrating density population behavior and the other one is showing a behavior of normalized density population that is $U/max(U)$. We investigate solution launching on the intermediate asymptotic forms calculating normalized population density $U/max(U)$ on every step of the problem's solving.

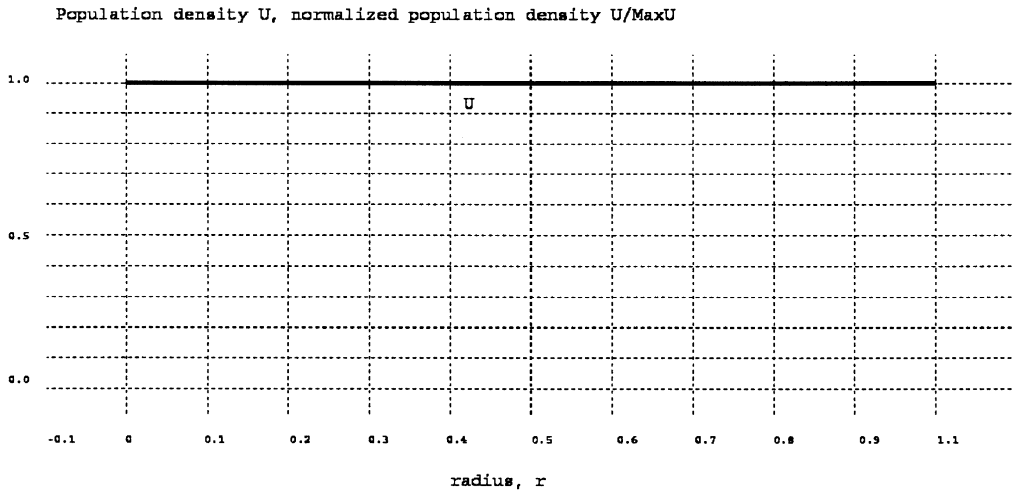

Fig. 1. A homogeneous distribution of functions.

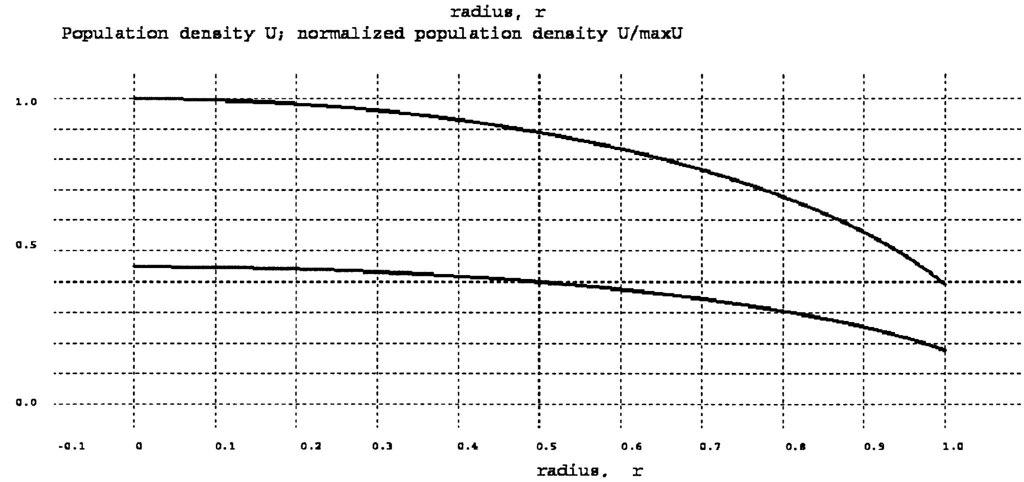

Fig. 2. Intermediate asymptotic form.

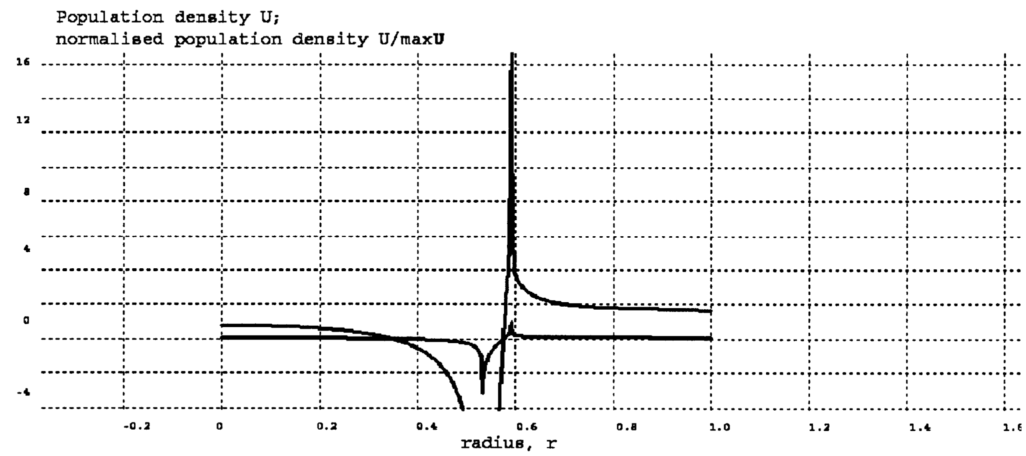

Fig. 3. “Saw” proves that solution is unstable.

3 Results of the computational experiments. Investigation of mixed problem solution’s behavior for the different functions k(U) and g(Q)

3.1 Investigation of solution’s behavior for $k(U) = U$ and $g(Q) = Q$

Let’s consider homogeneous distribution of U and Q functions across the area, that is $U(r, 0) = 1$, $Q(r, 0) = 1$. Bifurcation parameter $\nu = 0,5$ (Fig. 1).

Computational modelling shows that starting with $t = 1.5$ solution comes to the intermediate asymptotic form. That means normalized profile of density is not changing as time goes by. At the same time function is monotonously tending to zero (Fig. 2).

If bifurcation parameter ν is equal to 0.1 with the same initial and boundary conditions, then solution becomes unstable. Figure 3 can illustrate this. We can see the so-called “saw”, which proves that solution is unstable. Thus we can conclude that there is no solution of this problem in the late times.

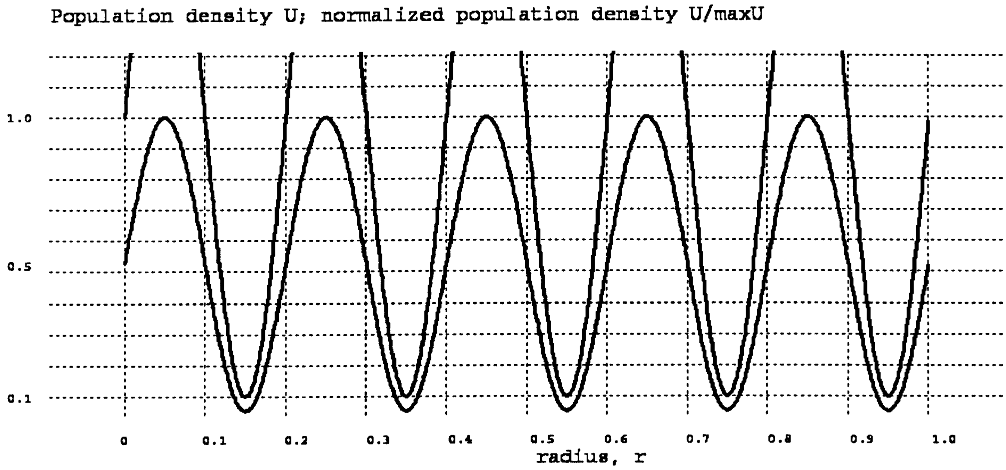

Fig. 4. Trigonometrical form.

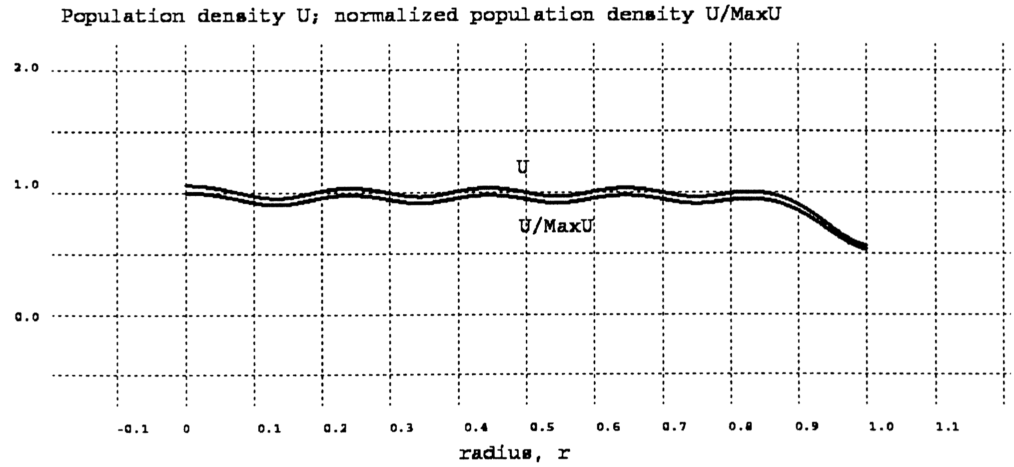

Fig. 5. Population density is moving to the intermediate asymptotic form.

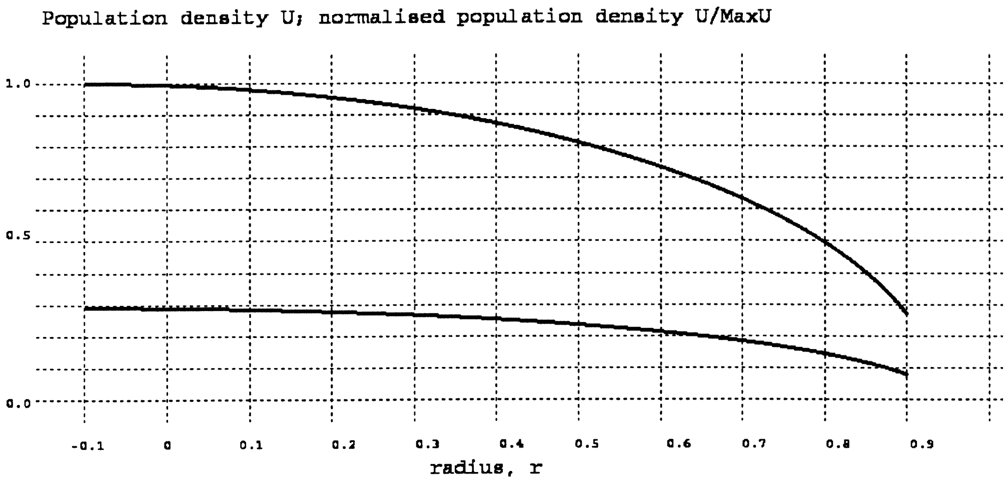

Fig. 6. Intermediate asymptotic form.

3.2 Population density — trigonometrical form

Let us take the following initial conditions:

$$\begin{aligned} &U(r,0) = 1 + 0,9\sin(10\pi r) \\ &Q(r,0) = 1 \\ &\nu = 0.5. \end{aligned} \tag{10}$$

As time goes by the system "quickly" forgets details of the initial conditions. In Figure 5, you can see transformation of sinusoid into the curve which form similar to another one from the example where $U = 1$. During the later point of time solution again comes to the intermediate asymptotic form, which can be proved by the diagram of the normalized density.

In the point of time $t = 4$ diagram is quite similar with case when $U(r,0) = r^3(1-r)^2$; Now we consider cases where we have changes in function $g(Q)$. Note, that initial conditions are still remaining the same.

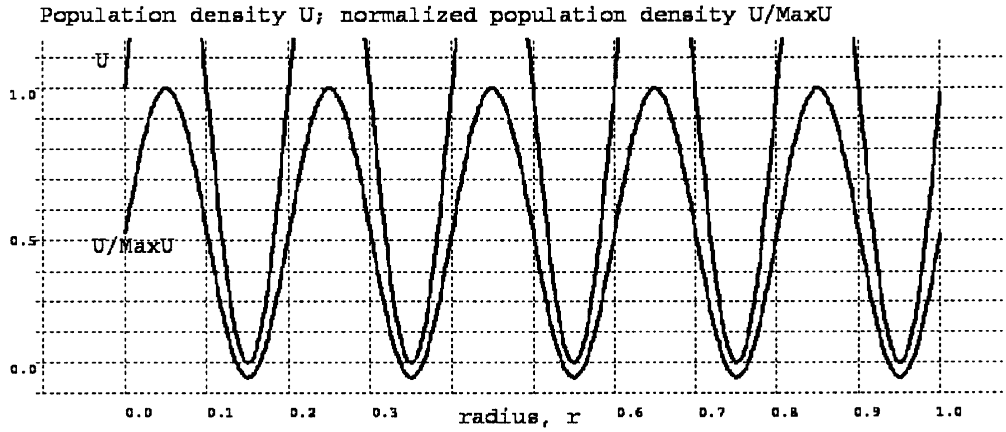

Fig. 7. Starting moment.

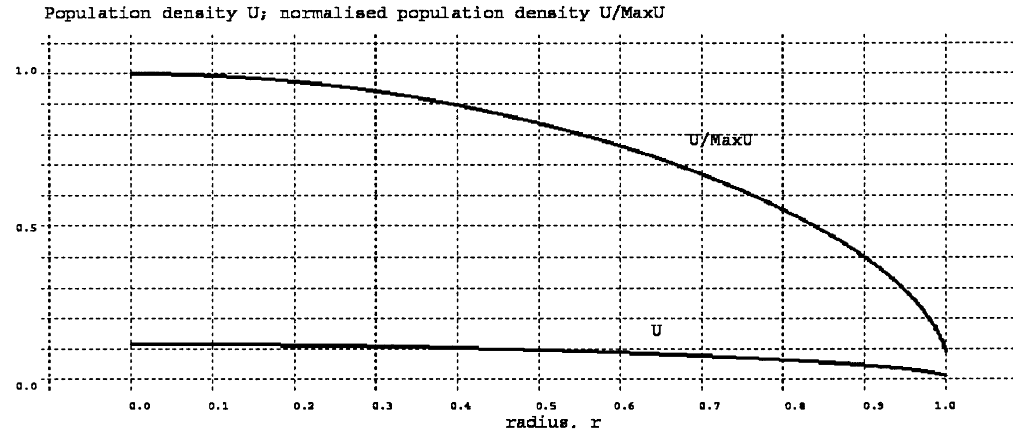

Fig. 8. Final result.

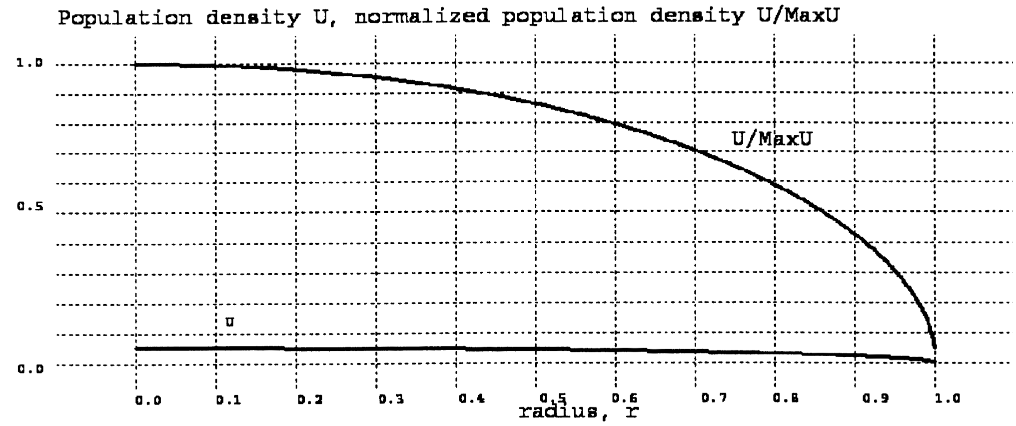

Fig. 9. Intermediate asymptotic form is similar to the form from previous case.

3.3 Investigation of the solution's behavior for the functions $k(U) = U$ and $g(Q) = Q^4$

Let $g(Q) = Q^4$, $k(U) = U$. Let us consider how solution can be changed if:

$$\begin{aligned} &U(r, 0) = 1 + 0,9 \sin(10\pi r) \\ &Q(r, 0) = 1 \\ &\nu = 0.5. \end{aligned} \tag{11}$$

The solution is faster tending to zero. However if we receive an intermediate asymptotic form (Fig. 8).

Now we consider $U(r, 0) = r^3(1 - r)^2$. As in previous case to the moment $t = 3$ solution has already come to the intermediate asymptotic form. So we can see again that initial conditions' details make no influence on the current situation. Normalized density is not changing in fact. We assume, that normalized density profile remains similar as in cases with another initial conditions. It differs from the profile which we received for $g(Q) = Q$, but similar for the different initial conditions.

3.4 Investigation of the solution's behavior for the functions $k(U) = e^U$ and $g(Q) = Q^4$

Since the aim of this work is finding out how the initial conditions ($k(U)$ and $g(Q)$)influence on the final solution, we can consider situation when $k(U) = e^U$. We can take the same three functions as initial conditions like in previous two cases.

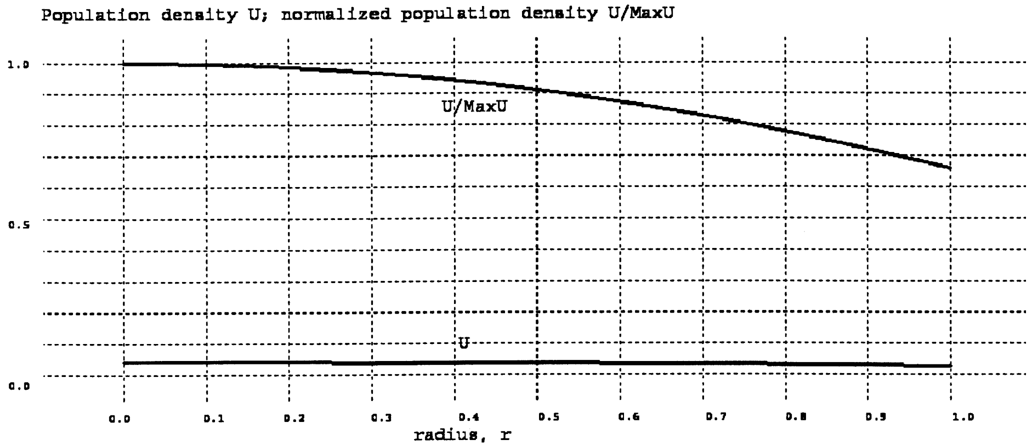

Fig. 10. Problem's solution faster comes to the intermediate asymptotic form.

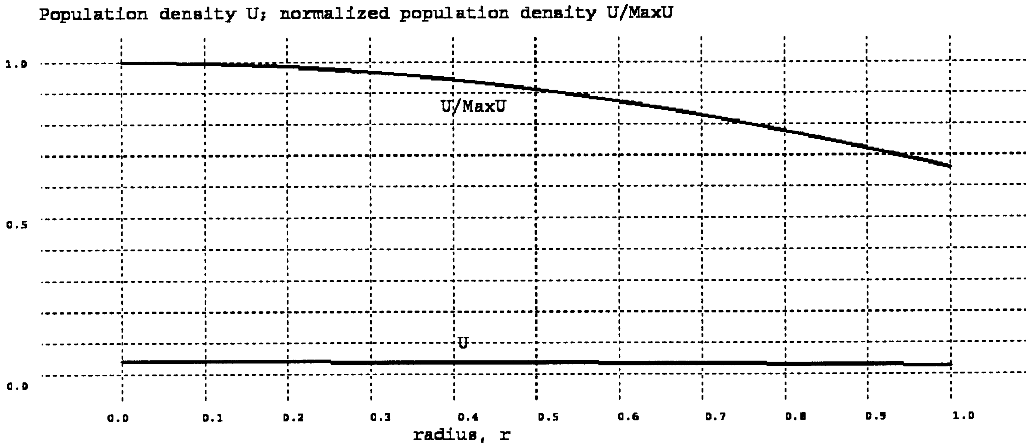

Fig. 11. Solution is situated higher then its analogues for other initial conditions.

So,

$$\begin{aligned} &U(r,0) = r^3(1-r)^2 \\ &Q(r,0) = 1 \\ &g(Q) = Q^4 \\ &k(U) = e^U \\ &\nu = 0.5. \end{aligned} \qquad (12)$$

It is clear that normalized density profile has changed in comparison with the case above. It has become flatter and the problem's solution faster comes to the intermediate asymptotic form (Fig. 10). The same situation can be seen if $U(r,0) = 1 + 0,9\sin(10\pi r)$.

The single difference consists in the following thing. If $t = 2$, solution is situated higher then its analogues for other initial conditions. Though normalized density has remained the same (Fig. 11).

4 Conclusion

We have made a computational simulation of the urban population density dynamics with the consideration of the housing quality modification. We have created a model consisting of a partial derivative equation of parabolic type and general differential equation. For the wide class of coefficients we have shown that problem's solution "quickly forgets" details of initial conditions and comes to the intermediate asymptotic form, which characterizes by the operator of the system only. In fact this means that urban system does not depend on the external circumstances and defines by the internal structure of the model. There are enormous variants of different conditions, which one can choose. So the investigation of all possible variations seems to be separate and rather difficult problem. Possibly, classification of the conditions should be made using data about real processes of urban formation and considering opinions of the geographers. The solution of this problem will be the next step to the understanding such problems as distribution of the population inside urban area, urban structure evaluation and other problems of urbanistics.

References

1. T. Puu, T.O.P., *Nonlinear Economic Dynamics* (Springer-Verlag, Berlin and Heidelberg, GmbH & Co. K, 1997)
2. Martin J. Beckmann, On spatial population dynamics, Complex Economic Phenomena in Time and Space in honour of Prof. Tonu Puu v. **18**(3), 421 (2003)
3. T. Puu, *Attractors, Bifurcations, and Chaos: Nonlinear Phenomena in Economics* (Springer-Verlag, Berlin and Heidelberg, GmbH & Co. K, July, 2003)
4. W.B. Zhang, *Synergetic Economics* (Springer-Verlag, Berlin and Heidelberg, GmbH & Co. KG, 1990)